Exact Analysis of Scaling and Dominant Attractors Beyond the Exponential Potential

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Abstract

By considering the potential parameter Γ as a function of another potential parameter λ[47], We successfully extend the analysis of two-dimensional autonomous dynamical system of quintessence scalar field model to the analysis of three-dimension, which makes us be able to research the critical points of a large number of potentials beyond the exponential potential exactly. We find that there are ten critical points in all, three points $P_{3,5,6}$ are general points which are possessed by all quintessence models regardless of the form of potentials and the rest points are closely connected to the concrete potentials. It is quite surprising that, apart from the exponential potential, there are a large number of potentials which can give the scaling solution when the function $f(\lambda)(=\Gamma(\lambda)-1)$ equals zero for one or some values of λ_* and if the parameter λ_* also satisfies the condition Eq.(16) or Eq.(17) at the same time. We give the differential equations to derive these potentials $V(\phi)$ from $f(\lambda)$. We also find that, if some conditions are satisfied, the de-Sitter-like dominant point P_4 and the scaling solution point P_9(or P_{10}) can be stable simultaneously but P_9 and P_{10} can not be stable simultaneity. Although we survey scaling solutions beyond the exponential potential for ordinary quintessence models in standard general relativity, this method can be applied to other extensively scaling solution models studied in literature[46] including coupled quintessence, (coupled-)phantom scalar field, k-essence and even beyond the general relativity case $H^2 \propto \rho_T^n$. we also discuss the disadvantage of our approach.

Keywords: Scaling Solution; Dark Energy; three-dimensional autonomous dynamical system; Cosmology.

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1 Introduction

Scalar fields had played an essential role in modern cosmology in the past semi-century. This assumed scalar field had been used for various purposes in different cosmological research aspects[1], such as to drive inflation, to explain a time variable cosmological "constant" and so on. Especially, after the discovery of the accelerating expansion of universe, it has gained another hotly discussion as the candidate for dark energy. There are so many scalar field dark energy models, such as quintessence model[2-13], non-canonical scalar field model (including K-essence[14-17], phantom[18-22], B-I scalar field[23-29] and so on) and coupled scalar field model[30-31]. There are also detailed studies on the multi-scalar field models which give an effective state equation w_{eff} passing through the phantom divide line ($w = -1$)[32-34]. Some of these multi-scalar field models[35-36] can always evolve onto the regime of scalar field dominance ${\lambda_{eff}}^2 > 3\gamma$ even if each field has too steep a potential to drive the accelerating expansion. For all of these scalar field models we mention above, the important thing is to choose different form of kinetic terms and different potentials from a fundamental physical motivation or directly from the observation. As are expected, these different scalar field models will give different cosmological evolutions, different evolutions of state equation w, different values of sound speed c_s^2 and different cosmological perturbation. So they can in principle be distinguished or excluded by the increasing observation data.

The phase-plane analysis of the cosmological autonomous system is an effective method to find the cosmological scaling and dominant attractor solutions. A phase-plane analysis of cosmologies containing a barotropic fluid and a scalar field with an exponential potential was presented[37]. Hao and Li studied the attractor solution of phantom scalar field with the exponential potential[38-39]. On the other hand, L.Amendola considered the case of coupled quintessence[31]. The case of phantom scalar field interacting with dark matter was also investigated[40-41]. Guo also investigated the properties of the critical points of multi-field model with an exponential potential[42-43] One may realize that the potentials investigated in all these papers are the exponential form. Disregarding the important roles of the exponential potential in higher-order or higher-dimensional gravity theories and string or kaluza-klein type models, the reason that why they are choosing the exponential potential may be that, only the exponential potential can give a two-dimension autonomous system. Since in this case the value of the parameter Γ equals 1 and then another parameter λ equals a constant(see Eq.(4) for the definition of parameters Γ, λ), so the system(see Eqs.(5-7) below) will reduce to the two-dimension autonomous system. However, authors also considered the more complicated case when λ is a dynamically changing quantity[44-46]. They applied the discussion of constant λ to this case and obtained the so-called "instantaneous" critical points. For example, if Γ is a constant(but does not equal one), saying $\Gamma = (n+1)/n$, the corresponding potential is the inverse power-law potential $V(\phi) = V_0\phi^{-n}$ with $n > 0$.

One of the critical points $(x_c, y_c) = (\lambda/\sqrt{6}, [1-\lambda^2/6]^{1/2})$ will become the "instantaneous" critical point $(x(N) = \lambda(N)/\sqrt{6}, y(N) = [1-\lambda(N)^2/6]^{1/2})$. When $\Gamma > 0$, $\lambda(N)$ will decrease toward zero, then the "instantaneous" critical points will eventually approach $x(N) \to 0$ and $y(N) \to 1$. This method is not exact here and obviously the critical point is not a true critical point. Recently, a solution of multiple-attractor in three-dimension autonomous system of the quintessential models was studied in literature[47]. After writing the parameter Γ as a function of λ, the author obtained a tracker solution which is different from those discovered before and found a solution of multiple-attractor. Here we will extend the idea to an arbitrary function $\Gamma(\lambda)$. We will find out all the critical points of the dynamical autonomous system, and then investigate the properties of the critical points and their cosmological implications in general. Regarding parameter Γ as a function of λ is a quiet efficient approach since we can investigate many quintessence models with different potentials. Giving a concrete form of function $\Gamma(\lambda)$ is equivalent to give a concrete form of potential $V(\phi)$ since we can in principle figure out the potential via the relation between parameter Γ and λ. What are the general properties of the critical points when we consider the three-dimension autonomous system? Does there also exists scaling solution when we consider any function of $\Gamma(\lambda)$? Among all the critical points which critical points are the critical points for all quintessence and which are only relative to the concrete potentials? In our paper, we will try to shed light on these issues. The paper is organized as follows: in Section 2 we present the theoretical framework and give the differential relation between the function $\Gamma(\lambda)$ and potential $V(\phi)$. We find out all the critical points and investigate their properties in Section 3. We try to give the cosmological implications of these critical points in section 4. We briefly display our conclusions in section 5.

2 Basic theoretical frame

We start with a spatially flat Friedman-Robertson-Walker universe containing a scalar field ϕ and a barotropic fluid (with state equation $p_b = w_b\rho_b$). To simply, we give the Einstein equations directly:

$$H^2 = \frac{\kappa^2}{3}[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_b] \tag{1}$$

$$\dot{H} = -\frac{\kappa^2}{2}[\dot{\phi}^2 + (1+w_b)\rho_b] \tag{2}$$

The motion equation of the scalar field ϕ is:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{3}$$

Following[48], we define the following dimensionless variables:

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \lambda = -\frac{V'}{V}, \Gamma = \frac{VV''}{V'^2} \tag{4}$$

Where $V' = dV(\phi)/d\phi, V'' = d^2V(\phi)/d\phi^2$. Using Eq.(4), Eqs.(1-3) can be rewritten in the following dynamical form[37, 46, 48]:

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_b)x^2 + (1+w_b)(1-y^2)] \tag{5}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_b)x^2 + (1+w_b)(1-y^2)] \tag{6}$$

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x \tag{7}$$

where $N = ln(a)$. Here we should emphasize that Eqs.(5-7) is not a dynamical autonomous system since the parameter Γ is unknown. However, if we consider Γ as a function of λ, namely

$$\Gamma(\lambda) = f(\lambda) + 1 \tag{8}$$

then Eq.(7) becomes:

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 f(\lambda)x \tag{9}$$

Hereafter, Eqs(5-6) and Eq.(9) are definitely a dynamical autonomous system. We will see that Γ as a function of λ can cover many quintessential potentials. The three-dimension autonomous system reduces to two-dimension autonomous systems when $f(\lambda) = 0$. In this case, the potential is the exponential form which has been completely studied in many literatures. When $f(\lambda)$ equals a nonvanishing constant f_λ, then the potential is proportional to $(c_1\phi + c_2)^{-1/f_\lambda}$, which is just the potential which has been considered as "instantaneous" critical points[48]. Generally speaking, we can analyze any explicit function. For some more complicated form, $\Gamma(\lambda) = 1 + \frac{1}{n} - \frac{n\sigma^2}{\lambda^2}$ corresponds to $V(\phi) = \frac{V_0}{[cosh(\sigma\phi)]^n}$, $\Gamma(\lambda) = 1 \pm \frac{\alpha}{\lambda^2}$ corresponds to $V(\phi) = V_0 e^{\pm\alpha\phi(\phi+\beta)/2}$, $\Gamma(\lambda) = 1 + \frac{2}{\sqrt{\lambda}}$ corresponds to $V(\phi) = V_0 e^{1/\phi}$. The form of $\Gamma(\lambda) = 1 + \frac{1}{\beta} + \frac{\alpha}{\lambda}$, which corresponds to $V(\phi) = \frac{V_0}{(\eta + e^{-\alpha\phi})^\beta}$, was considered as an interesting cosmological model where the universe can evolve from a scaling attractor to a de-Sitter-like attractor by introducing a possible mechanism of spontaneous symmetry breaking[47].

In the paper[47], the author gave an approach to obtain the potential $V(\phi)$ as follows: Since the potential $V(\phi)$ is only a function of the field ϕ, then the parameters λ and Γ can be written as a function of field: $\lambda = P(\phi), \Gamma = Q(\phi)$. If the inverse function of $P(\phi)$ exists, then we have:

$$\Gamma = Q(P^{-1}(\lambda)) \equiv \mathcal{F}(\lambda) \tag{10}$$

Using the definition of λ and Γ, V'' can be written as $V'' = \frac{V'^2}{V}\mathcal{F}(-\frac{V'}{V}) \equiv F(V, V')$. Let $h = V'$, then

$$\frac{dh}{dV} = \frac{1}{h}F(V, h) = \frac{h}{V}\mathcal{F}(-\frac{h}{V}) \tag{11}$$

Now Eq.(11) is a one-order differential equation of h and V. Figuring out $h(V)$, the potential can be solved from equation $V'(\phi) = h(V(\phi))$.

Here we introduce another easier approach to get the potential $V(\phi)$. We start with $\frac{d\lambda}{dV} = \frac{d\lambda}{d\phi}\frac{d\phi}{dV} = -\frac{d(V'/V)}{d\phi}\frac{1}{V'} = -\frac{1}{V'}\frac{V''V-V'^2}{V^2}$. Using the definition of λ and Γ, and the Eq.(8), we get a one-order differential equation of λ and V:

$$\frac{d\lambda}{dV} = \frac{\lambda}{V}f(\lambda) \tag{12}$$

Integrating out $\lambda = \lambda(V)$, using the definition of λ, then we have following differential equation of potential:

$$\frac{dV}{V\lambda(V)} = -d\phi \tag{13}$$

So Eq.(12) and Eq.(13) give the route to obtain the potential $V = V(\phi)$. As far as we know, there are too many investigations to the two-dimension autonomous system (where the potential is exponential) but have not general investigations to the dynamical properties of three dimensional autonomous system. It is maybe very interesting to consider this issue. We know that previous exactly analysis to the critical points of the quintessence model are based on a concrete form of potential (i.e., the exponential form). In this case it is not easy to distinguish which critical points are common to all the quintessence models and which are only related to the special potentials. In view of what we mention above, we will take a new route in next section to investigate the critical points of the autonomous system with an arbitrary function of $f(\lambda)$. Furthermore, the results can be easily applied to any other concrete potentials as long as they can be solved from Eqs.(12-13).

3 Critical Points and their Properties

It is easily seen from Eq.(9) that $\lambda = 0, x = 0$ or $f(\lambda) = 0$ can make $d\lambda/dN = 0$ respectively. The critical points listed in TABLE 1 can be found from the Eqs.(5,6,9) after setting $dx/dN = dy/dN = d\lambda/dN = 0$. The properties of each critical point are determined by the eigenvalues of the Jacobi matrix of the three-dimension autonomous system. For a general three-dimension autonomous system:

$$\begin{cases} \frac{dx}{dN} = f_1(x, y, \lambda) \\ \frac{dy}{dN} = f_2(x, y, \lambda) \\ \frac{d\lambda}{dN} = f_3(x, y, \lambda) \end{cases} \tag{14}$$

The function f_1, f_2 and f_3 are only the function of x, y, λ, no variable N and other variables, we call this dynamical system as autonomous system. If f_1, f_2 and f_3 are only a linear combination of x, y, λ, Eq.(14) is linear autonomous system. Its critical points (x_c, y_c, λ_c) can be found from the set of functions $f_1 = f_2 = f_3 = 0$. Obviously, Eqs.(5,6,9) is not a linear autonomous system. However, the local behavior of the nonlinear autonomous system near a critical point can be deduced by linearizing the nonlinear system about this point and be studied using the linear autonomous system analysis method. The properties of each critical point are determined by the eigenvalues of the Jacobi matrix $\mathcal{A}$, where

$$\mathcal{A} = \left[\begin{array}{lll} \partial f_1(x,y,\lambda)/\partial x & \partial f_1(x,xy,\lambda)/\partial y & \partial f_1(x,y,\lambda)/\partial \lambda \\ \partial f_2(x,y,\lambda)/\partial x & \partial f_2(x,y,\lambda)/\partial y & \partial f_2(x,y,\lambda)/\partial \lambda \\ \partial f_3(x,y,\lambda)/\partial x & \partial f_3(x,y,\lambda)/\partial y & \partial f_3(x,y,\lambda)/\partial \lambda \end{array} \right]_{(x_c,y_c,\lambda_c)} \tag{15}$$

For a hyperbolic critical point[1], if all the eigenvalues of $\mathcal{A}$ or the real part of these eigenvalues are negative, the critical point is stable. This is to say, as long as one of the eigenvalues or the real part of these eigenvalues is positive, the critical point must be unstable. However, if the critical point of nonlinear autonomous system is a nonhyperbolic point[2] and the rest of its eigenvalues having negative real part, the properties of this point can not be simply determined by linearization method and need to resort to other more complicated methods[50]. From TABLE 1, we can see that point P_4 is just this kind of point. In previous literatures[30, 31, 37, 51], the authors generally neglected this nonhyperbolic point when they met it. In fact this point also has the important cosmological implication as other critical points and should not be ignored. We will explore the properties of this nonhyperbolic point P_4 in our paper using the center manifold theorem[50] (The full analysis process is given in the Appendix). We list all the points and their properties in the following TABLE 1. Note that we have neglected the cases with $y < 0$ since the system is symmetric under the reflection $(\lambda, x, y) \to (\lambda, x, -y)$ and time reversal $t \to -t$.

[1] Actually "critical point" in this paper is also called the "equilibrium point" in mathematics or "fixed point" in some physical literatures. a hyperbolic critical(equilibrium) point is the critical(equilibrium) point which has no eigenvalues with zero real part.

[2] i.e., its eigenvalues exist zero value or have zero real parts.

	(λ_c, x_c, y_c)	eigenvalues	Stability
P_1	$(0,1,0)$	$3(1-w_b), 3, 0$	unstable node
P_2	$(0,-1,0)$	$3(1-w_b), 3, 0$	unstable node
P_3	$(0,0,0)$	$-3(1-w_b)/2, 3\gamma/2, 0$	saddle point
P_4	$(0,0,1)$	$-3, -3\gamma, 0$	stable node for $f(0) > 0$
P_5	$(\lambda_a, 0, 0)$	$-3(1-w_b)/2, 3\gamma/2, 0$	saddle point
P_6	$(\lambda_*, 0, 0)$	$-3(1-w_b)/2, 3\gamma/2, 0$	saddle point
P_7	$(\lambda_*, 1, 0)$	$-\sqrt{6}\lambda_*^2 df_*, 3(1-w_b), \frac{1}{2}(6-\sqrt{6}\lambda_*)$	saddle point
P_8	$(\lambda_*, -1, 0)$	$\sqrt{6}\lambda_*^2 df_*, 3(1-w_b), \frac{1}{2}(\sqrt{6}\lambda_* + 6)$	saddle point
P_9	$(\lambda_*, \frac{\sqrt{6}}{6}\lambda_*, \sqrt{1-\frac{1}{6}\lambda_*^2})$	$\frac{1}{2}(\lambda_*^2-6), \lambda_*^2-3\gamma, -\lambda_*^2\lambda_* df_*$	Eq.(16)
P_{10}	$(\lambda_*, \frac{\sqrt{6}\gamma}{2\lambda_*}, \frac{\sqrt{6\gamma(1-w_b)}}{2\lambda_*})$	$-3\lambda_*\gamma df_*, \frac{3}{4}(w_b-1)$ $\pm\frac{3\sqrt{(1-w_b)}}{4\lambda_*}\sqrt{24\gamma^2-(9\gamma-2)\lambda_*^2}$	Eq.(17)

Where $f(0)$ is the value of function $f(\lambda)$ at $\lambda = 0$, $df_* \equiv \frac{df(\lambda)}{d\lambda}|_{\lambda_*}$. We limit the range of $w_b(=\gamma-1)$ as $0 \le w_b < 1$, $w_b = 0$ for matter and 1/3 for radiation. λ_a means an arbitrary value and λ_* is the value which makes $f(\lambda_*) = 0$. So points P_{7-10} appear only if the function $f(\lambda)$ can be zero for one or more values of λ_*. Here we simply consider that only one value λ_* makes the function $f(\lambda)$ zero.

However, readers should keep in mind that, to make $d\lambda/dN = 0$ in Eq.(9), we let $\lambda = 0, x = 0$ and $f(\lambda) = 0$ separately, and then find out all the points listed in TABLE 1. But we do not consider one special case that $\lambda^2 f(\lambda) \neq 0$ and then $d\lambda/dN \neq 0$ when $\lambda = 0$. In this case, P_1 and P_2 are no more critical points. For example, the product $\lambda^2 f(\lambda) = \frac{V_0\alpha^2}{\Lambda} \neq 0$ even if $\lambda = 0$ for the potential $V(\phi) = V_0[\cosh(\alpha\phi) - 1] + \Lambda$. So the necessary condition for the existence of equilibrium points with $x \neq 0$ is $\lambda^2 f(\lambda) = 0$.

$\lambda_*^2 < 6$ is the condition for critical point P_9 to exist and Eq.(16) is the condition for P_9 to be a stable node.

$$\lambda_*^2 < 3\gamma \ and \ \lambda_* df_* > 0 \tag{16}$$

$\lambda_*^2 > 3\gamma$ is the condition for critical point P_{10} to exist and Eq.(17) is its stable condition.

$$\begin{array}{l} 24\gamma^2/(9\gamma-2) > \lambda_*^2 > 3\gamma \ and \ \lambda_* df_* > 0 \ \ for \ P_{10} \ being \ a \ stable \ node \\ \lambda_*^2 > 24\gamma^2/(9\gamma-2) \ and \ \lambda_* df_* > 0 \ \ for \ P_{10} \ being \ a \ stable \ spiral \end{array} \tag{17}$$

the density parameter of ϕ field and its equation of state are, respectively:

$$\Omega_\phi = x^2 + y^2 \tag{18}$$

$$w_\phi = \frac{x^2 - y^2}{x^2 + y^2} \tag{19}$$

In order to investigate the expansive behavior of scale factor a, we also represent the decelerating factor:

$$
\begin{aligned}
q &= -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}/H^2 = \frac{\sum(1+3w_i)\rho_i}{2\sum\rho_i} = \frac{1}{2}\sum(1+3w_i)\Omega_i \\
&= \frac{3}{2}[(1-w_b)x^2 - (1+w_b)y^2 + (w_b + \frac{1}{3})]
\end{aligned} \tag{20}
$$

We list the other properties of these critical points in TABLE 2.

	(λ_c, x_c, y_c)	w_ϕ	Ω_ϕ	decelerating factor(q)
P_1	$(0, 1, 0)$	1	1	2
P_2	$(0, -1, 0)$	1	1	2
P_3	$(0, 0, 0)$	Undefined	0	$(3w_b + 1)/2$
P_4	$(0, 0, 1)$	-1	1	-1
P_5	$(\lambda_a, 0, 0)$	Undefined	0	$(3w_b + 1)/2$
P_6	$(\lambda_*, 0, 0)$	Undefined	0	$(3w_b + 1)/2$
P_7	$(\lambda_*, 1, 0)$	1	1	2
P_8	$(\lambda_*, -1, 0)$	1	1	2
P_9	$(\lambda_*, \frac{\sqrt{6}}{6}\lambda_*, \sqrt{1 - \frac{1}{6}\lambda_*^2})$	$\lambda_*^2/3 - 1$	1	$\lambda_*^2/2 - 1$
P_{10}	$(\lambda_*, \frac{\sqrt{6}\gamma}{2\lambda_*}, \frac{\sqrt{6\gamma(1-w_b)}}{2\lambda_*})$	w_b	$3\gamma/\lambda_*^2$	$(3w_b + 1)/2$

4 Cosmological Implications

After giving all the critical points and their properties of the three-dimension autonomous system, we will investigate their cosmological implications. We will show some interesting results which have not been found previously in other literatures. Moreover, we will also response to the questions we have proposed in Section 1. Investigating three-dimension autonomous system instead of the two-dimension autonomous system can help us consider more potentials which can not be investigated via two-dimension autonomous system. Moreover, from the view of three-dimension system, we can gain a more deeply understanding than from the two-dimension system. For example, we will point out which critical points are the critical points for all quintessence and which are only relative to the concrete potentials. We can find from TABLE 1 and TABLE 2 that: Though the stability of Points $P_{1,2}$ does not depend on the form of concrete potentials, Points $P_{1,2}$ only exist when $\lambda^2 f(\lambda) = 0$ at $\lambda = 0$. Points $P_{3,5,6}$ always exist for all quintessence models and their stability are regardless of the form of concrete potentials. Point P_4 is also the critical point for all quintessence, but its stability depends on the form of concrete potentials. Points P_{7-10} and their properties are closely connected to the concrete potentials since the value of λ_* is determined by the form of $f(\lambda)$. points P_{7-10} are even inexistence if $f(\lambda) \neq 0$ for any λ.

Of all the points, only Points $P_{3,5,6}$ are independent of the function $f(\lambda)$. In fact, they have the same properties and can be considered as one point. They are saddle points which tell us that the barotropic fluid dominated solution ($\lambda_c = 0, x_c = 0, y_c = 0$) where $\Omega_\phi = 0$ is unstable. However, even though they are unstable, the phase space trajectories may evolve in the vicinity of the barotropic fluid dominated solution for a quite long time and then leaves this state to approach to the possible future attractor. However, if $\gamma = 0$, these points are found to be a stable attractor and can be used to alleviate the relic density problem in inflation model[37].

Four of the critical points ($P_{1,2}(\lambda_c = 0, x_c = \pm 1, y_c = 0)$ and $P_{7,8}(\lambda_c = \lambda_*, x_c = \pm 1, y_c = 0)$) are all unstable nodes, which correspond to the solutions where the universe is dominated by the kinetic energy of the scalar field ($\Omega_\phi = 1$) with a stiff equation of state ($w_\phi = 1$).

In fact, we can conclude above results with one brief sentence (see TABLE 1): *all the critical points with y_c being zero are not stable points.* It tells us that, under the potential we considered here, the cosmological solution with the potential energy eventually evolving to zero will never be the final state of our universe. This is a quite interesting result since we know that the universe will never undergo a regime of accelerating expansion if there is no potential energy in quintessence models.

Therefore, there are only three critical points $P_{4,9,10}$ which correspond to possible late-time attractor solutions. We will study their properties and cosmological implications in more detail.

Points $P_{4,9}$ are both scalar field dominated solutions with $\Omega_\phi = 1$. Comparing with point P_4, P_9 is the well-known scalar field dominated solution which exists for $\lambda_*^2 < 6$. TABLE 1 has shown that this scalar field dominated solution is a later-time attractor in the presence of a barotropic fluid if we have $\lambda_*^2 < 3\gamma$ *and* $\lambda_* df_* > 0$. This solution will give an accelerating universe if $\lambda_*^2 < 2$ *and* $\lambda_* df_* > 0$. For example, $f(\lambda) = \frac{1}{n} - \frac{n\sigma^2}{\lambda^2}$ corresponds to $V(\phi) = \frac{V_0}{[cosh(\sigma\phi)]^n}$. Obviously we have $\lambda_* = \pm|n\sigma|$ and $df_* = \frac{2n\sigma^2}{\lambda_*^3}$. The scalar field dominated solution with potential $V(\phi) = \frac{V_0}{[cosh(\sigma\phi)]^n}$ is a late-time attractor if $n^2\sigma^2 < 3\gamma$ *and* $\frac{2n\sigma^2}{\lambda_*^2} > 0$. In addition, this solution admits an accelerating expansion of universe if $n^2\sigma^2 < 2$ *and* $\frac{2n\sigma^2}{\lambda_*^2} > 0$. Noted that the point P_9 means two stable critical points ($\lambda_c = \pm|n\sigma|, x_c = \frac{\pm\sqrt{6}}{6}|n\sigma|, y_c = \sqrt{1 \mp \frac{1}{6}n^2\sigma^2}$) in this case .

P_{10} is the scaling solution where neither the scalar field nor the barotropic fluid entirely dominates the universe. P_{10} is a stable node for $24\gamma^2/(9\gamma - 2) > \lambda_*^2 > 3\gamma$ *and* $\lambda_* df_* > 0$ and a stable spiral for $\lambda_*^2 > 24\gamma^2/(9\gamma - 2)$ *and* $\lambda_* df_* > 0$. So P_9 and P_{10} can not be stable simultaneously. The scaling solution has drawn a lot of attentions since it can alleviate the coincidence problem of dark energy. Many potentials have been proposed to give a scaling evolution regime[35, 36, 52-65]. Here we give a sufficient condition for a potential to possess a scaling solution, that is, *as long as $f(\lambda)$ equals zero for one or more values of*

$\lambda(=\lambda_*)$ *and these* λ_* *also satisfy Eq.(17), then there must exist a scaling solution with* $\Omega_\phi = 3\gamma/\lambda_*^2$. Obviously many potentials which satisfy this condition exist, such as the potential $V(\phi) = \frac{V_0}{[cosh(\sigma\phi)]^n}$ which corresponds to $f(\lambda) = \frac{1}{n} - \frac{n\sigma^2}{\lambda^2}$, the potential $V(\phi) = \frac{V_0}{(\eta+e^{-\alpha\phi})^\beta}$ which corresponds to $f(\lambda) = \frac{1}{\beta}+\frac{\alpha}{\lambda}$ and so on. Our condition includes the potentials in Ref[49] where the authors found that every positive and monotonous potential which was asymptotically exponential yielded a scaling solution. Our result is also not contradiction to the statement in literature[66, 67] where they assumed a scaling solution like P_{10} and found the potential was unique the exponential form. This exponential potential is explicitly figured out from the assumption and the evolution of universe with this potential is always the scaling solution(see Eq.(18) in literature[66]) while P_{10} being a stable point means that all the evolution of the universe with a class of potentials which satisfy Eq.(17) will all approach the scaling solution finally. It is just an asymptotic behavior at late time. Unfortunately, for the scaling solution of P_{10}, the state equation of dark energy w_ϕ equals w_m and therefore there does not exist the accelerating expansion if w_m is larger than zero. However, authors had obtained the exact quintessence potential $V(\phi) = \frac{1-w_\phi}{2}\rho_{\phi 0}[\sqrt{\frac{\Omega_{m0}}{\Omega_{\phi 0}}}sinh(\frac{3(w_m-w_\phi)}{2\sqrt{3(1+w_\phi)}}\frac{\phi-\phi_{in}}{m_{pl}})]^{-2(1+w_\phi)/(w_m-w_\phi)}$, which admited a scaling solution with $w_\phi \neq w_m$ *and* $\Omega_\phi \neq 0$[66]. With this potential, in principle, we can obtain a scaling solution with an accelerating expansion of the universe.

Finally, let us consider the point P_4, which is a de-Sitter-like dominant attractor with $\Omega_\phi = 1$ *and* $w_\phi = -1$. The condition for P_4 being a stable point is that the value of $f(\lambda)$ when $\lambda = 0$ must be larger than zero(i.e., $f(0) > 0$, see appendix for details). So generally speaking, P_4 and P_9(or P_{10}) also can not be stable simultaneously. However, there may exist the possibility for some potentials that their values at $\lambda = 0$ is larger than zero but equals zero for some others λ_* ($\lambda_* \neq 0$), then this region of λ in the phase space of the three dynamical autonomous system will lie in the basin of the attractor P_{10}. That means, in this case, there can exist two stable critical points simultaneously, but this is not to say that the universe can evolve continuously from one stable critical point to another one. Based on this fact, the author proposed a scenario of universe which could evolve from a scaling attractor to a de-Sitter-like attractor by introducing a field whose value changed a certain amount in a short time[47]. In fact, we can also obtain these two asymptotical evolutions if the potential $V(\phi)$ can be approximated to two different potentials when ϕ evolves to different range, one admits the scaling solution and another one admits the de-Sitter-like solution[53, 58, 61]. For these potentials, the exit of the cosmological evolution from one attractor solution to another attractor is quite natural, but the explanation of why we have these special potentials is not quite natural.

5 Conclusion

In this paper, we extend the autonomous dynamical system analysis of the canonical scalar field from 2-D to three-dimension by considering the potential parameter Γ as a function of another potential parameter λ. There are ten critical points in all: three of these points ($P_{3,5,6}$) are general points which are possessed by all quintessence models regardless of the form of potentials and the rest points, with their existence or/and stability, are closely connected to the concrete potentials. We surprisingly find that, apart from the exponential potential, there are a large number of potentials which can give the scaling solution when the function $f(\lambda)(=\Gamma(\lambda)-1)$ equals zero for one or some values of λ and the parameter λ satisfies the condition Eq.(16) or Eq.(17) at the same time. We give the explicit expression to derive these potentials $V(\phi)$ from $f(\lambda)$. We find that, if some conditions are satisfied, the de-Sitter-like dominant point P_4 and the scaling point P_9(or P_{10}) can simultaneously be stable, but P_9 and P_{10} can not be stable at one time. As we have seen, the autonomous dynamical systems analysis is a very powerful tool which helps us extract useful cosmological information without solving the complicated background equations. Our method extends the analysis from two-dimensional autonomous dynamical system to three-dimension, which makes us be able to research a large number of potentials beyond the exponential potential. This method is quite effective and may be applied to a broad class of dark energy models studied in literature[46], including coupled quintessence, (coupled-)phantom scalar field, k-essence and even generalized background $H^2 \propto \rho_T^n$.

However, we should point out that our approach also has its drawbacks. First, as we have mentioned above: the approach can not be applied for the potentials for which the function $\Gamma = VV''/(V')^2$ can not be written as an explicit function of the variable λ. Second, the variable λ is undefined if the potentials vanish at its minimum, so the approach can not be applied for the potentials which vanish at its minimum. But, in despite of the second problem, it is actually not a fatal drawback. On one hand, the minimum of a potential is always associate with the late-time cosmological dynamics(future attractors). It is quite easy to discuss this special equilibrium point separately if we know a given potential has minimum(It is usually not difficult to find out the minimum of a given function). On the other hand, we can still use our approach to analyze all the critical points P_{1-10} since λ is well-defined around these critical points. We take the potential $V(\phi) = V_0[cosh(\alpha\phi) - 1]$ for example, this potential has a minimum value 0 at $\phi = 0$(λ has no definition at $\phi = 0$). We can investigate the critical point corresponding to this minimum separately. The explicit function about this potential is $f(\lambda) = \frac{1}{2}(\frac{\alpha^2}{\lambda^2} - 1)$. Obviously, the point corresponding to the potential's minimum does not appear in the Table 1. We can still discuss the properties of the critical points P_{1-10} even if the variable λ has no definition sometime.

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Appendix

In section 3, we pointed out that if the eigenvalues of Jacobi matrix had one or more eigenvalues with zero real parts while the rest of the eigenvalues had negative real parts, then linearization fails to determine the stability properties of this critical point. From TABLE 1 we realize that point P_4 is just such point, so in this Appendix we will show you that how we get the stable condition of P_4 from the center manifold theorem. The point P_4 is $(\lambda_c = 0, x_c = 0, y_c = 1)$ and its three eigenvalues are $(0, -3, -3(1 + w_m))$. Firstly, we transfer P_4 to P_4' $(\lambda_c = 0, x_c = 0, Y_c = y_c - 1 = 0)$ for convenience. In this case, Eqs.(5-7) can be rewritten as:

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2 f(\lambda)x \tag{21}$$

$$\frac{dx}{dN} = -3x+\frac{1}{2}\sqrt{6}\lambda+\frac{1}{2}\sqrt{6}\lambda Y^2+\sqrt{6}\lambda Y+\frac{3}{2}x^3(1-w_m)-\frac{3}{2}(1+w_m)xY^2-3(1+w_m)xY \tag{22}$$

$$\frac{dY}{dN} = -3(1+w_m)Y-\frac{1}{2}\sqrt{6}\lambda x(Y+1)+\frac{3}{2}(1-w_m)x^2Y-\frac{3}{2}Y^3-\frac{3}{2}(3+w_m)Y^2+\frac{3}{2}(1-w_m)x^2 \tag{23}$$

Noted that $\{\lambda, x, Y\}$ in Eqs.(21-23) are very small variables around point $(\lambda_c = 0, x_c = 0, Y_c = 0$. So the function $f(\lambda)$ in Eq.(21) should be taken the Taylor series in λ: $f(\lambda) = f(0) + f^1(0)\lambda + \frac{f^2(0)}{2!}\lambda^2 + ...$, where $f^n(0)$ is the value of $\frac{d^n f(\lambda)}{d\lambda^n}$ when $\lambda = 0$.

We can write down the Jacobi matrix $\mathcal{A}$ of dynamical system Eqs.(21-23):

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2}\sqrt{6} & -3 & 0 \\ 0 & 0 & -3(1+w_m) \end{bmatrix} \tag{24}$$

The eigenvalues of $\mathcal{A}$ and the corresponding eigenvectors are:

$$\{0, \ [1, \ \frac{\sqrt{6}}{6}, 0]\}; \qquad \{-3, \ [0,1,0]\}; \qquad \{-3(1+3w_m), \ [0,0,1]\} \tag{25}$$

Let $\mathcal{M}$ be a matrix whose columns are the eigenvectors of $\mathcal{A}$, then we can write down $\mathcal{M}$ and its inverse matrix $\mathcal{T}$:

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{T} = \mathcal{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{26}$$

Using the similarity transformation $\mathcal{T}$ we can transform $\mathcal{A}$ into a block diagonal matrix, that is,

$$\mathcal{T}\mathcal{A}\mathcal{T}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3(1+w_m) \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix} \tag{27}$$

where all eigenvalues of $\mathcal{A}_1$ have zero real parts and all eigenvalues of $\mathcal{A}_2$ have negative real parts. We put a change of variables:

$$\begin{bmatrix} \lambda' \\ x' \\ Y' \end{bmatrix} = \mathcal{T} \begin{bmatrix} \lambda \\ x \\ Y \end{bmatrix} = \begin{bmatrix} \lambda \\ -\frac{\sqrt{6}}{6}\lambda + x \\ Y \end{bmatrix} \tag{28}$$

Then we can rewrite the dynamical system Eqs.(21-23) in the form of new variables:

$$\frac{d\lambda'}{dN} = \frac{d\lambda}{dN} = f_1(\lambda', x', Y') \tag{29}$$

$$\frac{dx'}{dN} = -\frac{\sqrt{6}}{6}\frac{d\lambda}{dN} + \frac{dx}{dN} = f_2(\lambda', x', Y') \tag{30}$$

$$\frac{dY'}{dN} = \frac{dY}{dN} = f_3(\lambda', x', Y') \tag{31}$$

the detail forms of $f_1(\lambda', x', Y')$, $f_2(\lambda', x', Y')$, $f_3(\lambda', x', Y')$ are easily obtained after we substitute the transformation $\lambda = \lambda'$, $x = \frac{\sqrt{6}}{6}\lambda' + x'$ and $Y = Y'$ into the right hand of Eqs.(21-23). According to the center manifold theorem, the stable condition of dynamical system Eq.(21-23), i.e., the stability of P_4 will be finally determined by the following simple reduced system:

$$\frac{d\lambda'}{dN} = \frac{d\lambda}{dN} = -\lambda'^3 f(0) = -\lambda^3 f(0) \tag{32}$$

$f(0)$ is the value of function $f(\lambda)$ at $\lambda = 0$. This simple one-dimensional dynamical system Eq.(32) is stable if $f(0) > 0$.

So we conclude that P_4 is a stable de-Sitter-like dominant attractor when $f(0) > 0$, just as shown in TABLE 2.

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